GENERAL CRITERIA FOR CURVES TO BE SIMPLE

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ABSTRACT. We extend previous results for parametrized curves in euclidean space to be simple. The new condition depends as before on Ahlfors' Schwarzian and considers a *conformal metric* on a given interval and the new diameter. We derive some applications, among which we find Becker type conditions that depend on a pre-Schwarzian.

1. INTRODUCTION

The purpose of this paper is to extend results in [5], where the use of Sturm comparison and Ahlfors' Schwarzian for curves led to sufficient conditions for parametrized curves in euclidean space to be simple. In many cases, the condition was sharp. By considering a "conformal metric" on an interval, we derive here a more general condition of the same type that takes into account the modified diameter of the interval. The theorem fills in the gaps when the former condition was not optimal. In addition, suitable choices of the conformal factor give rise to criteria that depend on a pre-Schwarzian derivative, and analogues of criteria for holomorphic mappings in the disk due to Ahlfors, Becker, and Epstein [2], [3], [9].

We begin with a brief account on Ahlfors' Schwarzian for curves. In [1] the author generalizes the Schwarzian to cover $f:(a,b)\to\mathbb{R}^n$ by separately defining analogues of the real and imaginary parts $\mathrm{Re}\{Sf\},\mathrm{Im}\{Sf\}$ of the Schwarzian of a locally injective mapping f. For parametrized curves with $f'\neq 0$ he defined

$$S_1f=\frac{\langle f',f'''\rangle}{|f'|^2}-3\frac{\langle f',f''\rangle^2}{|f'|^4}+\frac{3}{2}\frac{|f''|^2}{|f'|^2}\,, \tag{1.1}$$

and

$$S_2f=\frac{f'\wedge f'''}{|f'|^2}-3\frac{\langle f',f''\rangle}{|f'|^4}f'\wedge f''\,, \tag{1.2}$$

respectively. Here,$\langle\,,\,\rangle$ denotes the standard inner product, and for $\vec a,\vec b\in\mathbb{R}^n$, $\vec a\wedge\vec b$ is the antisymmetric bivector with components $(\vec a\wedge\vec b)_{ij}=a_ib_j-a_jb_i$ and norm $\left[\sum(a_ib_j-a_jb_i)^2\right]^{1/2}$. Ahlfors indicated that he was led to these seemingly esoteric

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definitions by a direct identification of $\mathrm{Re}\{z\bar{\zeta}\}$ with the inner product $\langle z,\zeta\rangle$ of the 2-dimensional vectors z,ζ and the far from obvious identification of $\mathrm{Im}\{z\bar{\zeta}\}$ with the corresponding $z\wedge\zeta$. For the purpose of injectivity, only $S_1 f$ has played a role so far. In [5] a simpler form was obtained for $S_1 f$ in the form

$$S_1 f = \left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2 + \frac{1}{2}v^2k^2\,,$$

where $v=|f'|$ and k is the curvature. Thus, if $s=s(x), x\in(a,b)$, stands for the arclength parameter, then

$$S_1 f = Ss(x) + \frac{1}{2}v^2k^2\,, \tag{1.3}$$

where $Sh=(h''/h')'-(1/2)(h''/h')^2$ is the usual Schwarzian.

Since the operators introduced by Ahlfors are Möbius invariant (see [1], [5]), one can include curves into $\mathbb{R}^n\cup\{\infty\}$. Another important property that will be used is that of a chain rule, which states that under a change of parameter $x=h(t)$,

$$S_1(f\circ h) = [(S_1 f)\circ h]\,(h')^2 + Sh\,.$$

The combination of these properties together with comparison techniques from the Sturm theory has resulted in important applications of the S_1 operator in questions regarding the injectivity of the conformal immersion of planar domains into higher dimensional euclidean space [6], [7], [4], [12].

The following results were established in [5]:

Theorem A. Let $p=p(x)$ be a continuous function defined on an interval $I\subset\mathbb{R}$ with the property that no non-trivial solution of

$$u''+pu=0 \tag{1.4}$$

has more than one zero. Let $f: I\to\mathbb{R}^n\cup\{\infty\}$ be a C^3 curve with $f'\neq 0$ on I. If $S_1 f\le 2p$ then f is injective on I.

Suitable choices of functions p on the interval $I=(-1,1)$ were shown to render analogues of the classical injectivity criteria due to Nehari [11]. For a coefficient function p such as in the theorem, the differential equation (1.4) is called *disconjugate*. By considering $p=\pi^2/\delta^2$ on an interval I of length δ we obtain from Theorem A the important corollary that if

$$S_1 f \le \frac{2\pi^2}{\delta^2} \tag{1.5}$$

then f is injective. This choice of function p satisfies the hypothesis of the theorem since one can arrange for a suitable trigonometric function to be a non-vanishing solution of $u''+(\pi^2/\delta^2)u=0$ on I.

In theorem B below, the interval was normalized to be $(-1,1)$. The analysis was simplified with the additional assumption of symmetry of the function p. The function $F:(-1,1)\to\mathbb{R}$ was defined by the conditions $SF=2p$ and $F(0)=0, F'(0)=1, F''(0)=0$.

Theorem B. Let $f:(-1,1)\to\mathbb{R}^n\cup\{\infty\}$, $f'\neq 0$, satisfy $f(0)=0, |f'(0)|=1, f''(0)=0$ and suppose that $S_1 f\leq 2p$ for some even function p for which (1.4) is disconjugate in $(-1,1)$. Then

(a) $|f'(x)|\leq F'(|x|)$ on $(-1,1)$ and f admits a (spherically) continuous extension to $[-1,1]$.

(b) If $F(1)<\infty$ then f is one-to-one on $[-1,1]$ and $f([-1,1])$ has finite length.

(c) If $F(1)=\infty$ then either f is one-to-one on $[-1,1]$ or, up to a rotation, $f=F$.

2. Main Result

We first establish an injectivity condition for parametrized curves that parallels Theorem A.

Theorem 2.1. *Let $\psi=\psi(x)$ be a C^2 function on an interval $I\subset\mathbb{R}$, and let*

$$\delta=\int_I e^{\psi}dx\,.$$

Let $f:I\to\mathbb{R}^n\cup\{\infty\}$ be C^3 with non-vanishing f'. If

$$S_1 f\leq \psi''-\frac{1}{2}(\psi')^2+e^{2\psi}\frac{2\pi^2}{\delta^2}\,, \tag{2.1}$$

then f is injective on I.

Proof. Fix $x_0\in I$ and let

$$F(x)=\int_{x_0}^{x} e^{\psi(t)}dt\,. \tag{2.2}$$

The function F maps $I=(a,b)$ in a one-to-one manner onto an interval J of total length δ. It will be shown that (2.1) implies (1.5) for the curve $g=f\circ h$, where $h=F^{-1}$.

From the chain rule we see that

$$Sh=-(SF)(h')^2=-[\psi''-\frac{1}{2}(\psi')^2](h')^2\,,$$

and therefore

$$S_1 g=(S_1 f\circ h)(h')^2+Sh\leq\frac{2\pi^2}{\delta^2}\,.$$

This proves that g is injective on J, hence f is injective on I. □

We claim that Theorem A follows from Theorem 2.1 by choosing ψ adequately. It is well known that a locally injective function $G:I\to\mathbb{R}\cup\{\infty\}$ has $SG=2p$ if and only if

$$G=\frac{u_1}{u_2}$$

for a pair u_1, u_2 of linearly independent solutions of (1.4) [8]. On the other hand, given a solution u, variation of parameters gives a second, linearly independent solution of the form $u(x)\int^x u^{-2}(t)dt$. Consequently,

$$G(x) = \int^x u^{-2}(t)dt$$

is always a function with $SG = 2p$, with the zeros of u mapped to the point at infinity. The action of the Möbius transformations $T(x) = (Ax+B)/(Cx+D)$, $AD - BC \neq 0$, gives rise to all other functions $H = T(G)$ with $SH = 2p$ and all other solutions of (1.4), $v = (H')^{-1/2}$.

Suppose now that p satisfies the hypothesis of Theorem A. Any function with $SG = 2p$ will be injective on I (see, *e.g.*, [8]), and thus $J = G(I)$ is a non-overlapping interval on $\mathbb{R} \cup \{\infty\}$. Therefore, the set $\mathbb{R} \cup \{\infty\} \backslash J$ contains at least one point, and by choosing a suitable Möbius shift $H = T(G)$, we may assume that $\infty \notin J$. Hence $u = (H')^{-1/2}$ is a solution of (1.4) that is non-vanishing on I. We now let

$$e^{\psi} = u^{-2}\,,$$

which gives that

$$SH = 2p = \psi'' - \frac{1}{2}(\psi')^2\,.$$

This shows that Theorem A can always be obtained from Theorem 2.1.

We now analyze under what circumstances Theorem 2.1 improves Theorem A, that is, when is it possible to choose $\psi = -2\log u$ for a non-vanishing solution of (1.4) so that $\delta < \infty$. We distinguish two cases. Suppose first that the interval J, as chosen above, is the entire real line $\mathbb{R}$. This means that

$$\int_a u^{-2}(t)dt = \int^b u^{-2}(t)dt = \infty\,,$$

which is equivalent to saying that u is principal at both endpoints of I. Any other solution of (1.4) that does not vanish will be a constant multiple of u [10]. In this case, Theorem 2.1 will not improve Theorem A (and corresponds to part (c) in Theorem B).

Suppose now that J is a proper subinterval of $\mathbb{R}$. This allows for a second Möbius shift so that J is a bounded interval. Hence, there exists a nowhere vanishing solution u of (1.4) that is not principal at either endpoint a, b. The choice $\psi = -2\log u$ produces a finite diameter, and Theorem 2.1 improves Theorem A exactly by the last term in (2.1).

An interesting point is whether, in this last case, there exists an "optimal" choice of function ψ, meaning a choice of a bounded interval J so that the term

$$\Lambda = \Lambda_F = e^{2\psi}\frac{2\pi^2}{\delta^2}$$

is maximal for all $x \in I$. We analyze the effect on Λ of Möbius shifts $G = T(F)$ that map $J = F(I) = (\alpha, \beta)$ to another bounded interval. It is readily seen that $\Lambda_F = \lambda_G$ when T is affine. If T is not affine, then up to an affine change, T is an inversion of the form

$$T(y) = \frac{1}{y - y_0}\,,$$

for some $y_0 \notin \overline{J}$. A direct calculation shows that

$$\Lambda_G = \mu^2 \Lambda_F\,,$$

where

$$\mu = \frac{(y_0 - \alpha)(y_0 - \beta)}{(y_0 - F(x))^2}\,.$$

The extreme values of μ are the reciprocal quantities

$$\left|\frac{y_0 - \alpha}{y_0 - \beta}\right| \quad , \quad \left|\frac{y_0 - \beta}{y_0 - \alpha}\right| ,$$

and are attained when $F(x)$ is an endpoint of J. The values of μ stay close to 1 for relatively large y_0, but can vary significantly when y_0 is close to an endpoint of J. In summary, among all functions ψ for which $\psi'' - (1/2)(\psi')^2 = 2p$ there is no choice that maximizes the term Λ for all $x \in I$.

We finally analyze the sharpness of Theorem 2.1. For given ψ we let $2p = \psi'' - (1/2)(\psi')^2$. Since $u = e^{-\psi/2}$ is a non-vanishing solution of $u'' + pu = 0$, we see that any solution of this equation can vanish at most once on I. We will say that Theorem 2.1 is of *infinite type* if

$$\int_a e^{\psi} dx = \int^b e^{\psi} dx = \infty\,. \tag{2.3}$$

Hence $\delta = \infty$ and the equation $u'' + pu = 0$ admits a non-vanishing solution that is principal at both endpoints of I.

We will say that Theorem 2.1 if of *finite type* if at least one of the integrals in (2.3) is finite. As we have seen, in this case it is possible to modify ψ without changing $2p = \psi'' - (1/2)(\psi')^2$ in a way that both integral in (2.3) are finite.

We claim that Theorem 2.1 is always sharp, in the following senses. First, there exists a curve satisfying the hypothesis which fails to be injective in the closed interval $\overline{I}$. It is also sharp in the sense that given any $\epsilon = \epsilon(x) \geq 0$ defined on I which is not identically zero, there exists a non-injective $f : I \to \mathbb{R}^n \cup \{\infty\}$ with

$$S_1 f \leq \psi'' - (1/2)(\psi')^2 + e^{2\psi}\frac{2\pi^2}{\delta^2} + \epsilon\,.$$

To establish these claims we consider the function F defined by (2.2). If we are in the case of infinite type then $F(a) = F(b)$ is the point at infinity, hence F fails to be injective on $\overline{I}$. In the case of finite type, we can assume that ψ is chosen to produce $\delta < \infty$. Choose $x_0 \in I$ so that $F(b) = -F(a) = \delta/2$, and let

$$\phi(x) = \tan\left(\frac{\pi}{\delta}F(x)\right) .$$

Then ϕ is an increasing function mapping I onto $\mathbb{R}$, with $\phi(a) = \phi(b)$ equal to the point at infinity. Thus ϕ is not injective on $\overline{I}$. Furthermore,

$$S\phi = \psi'' - (1/2)(\psi')^2 + e^{2\psi}\frac{2\pi^2}{\delta^2} .$$

We call the functions F, ϕ *extremals* for Theorem 2.1 depending on the type. In the last section, we will show them to be unique up to Möbius transformations.

The second claim follows in both cases from the following theorem.

Theorem 2.2. *Let $H : I \to \mathbb{R}$ be a C^3 function with $H' > 0$ and $SH = 2q$, and suppose that $H(I) = \mathbb{R}$. If $\epsilon = \epsilon(x) \geq 0$ is not identically zero then the differential equation*

$$v'' + (q + \epsilon)v = 0 \tag{2.4}$$

admits a non-trivial solution with at least two zeros.

Proof. Fix $x_0 \in I$ and consider the solution of (2.4) with

$$v(x_0) = u(x_0) \quad , \quad v'(x_0) = u'(x_0) ,$$

where $u = (H')^{-1/2}$. For $y \in \mathbb{R}$ let

$$w(y) = \frac{v}{u}\left(H^{-1}(y)\right) .$$

Then

$$w''(y) = -(\epsilon u^4)w(y) ,$$

with ϵu^4 evaluated at $H^{-1}(y)$. Furthermore, $w(y_0) = 1, w'(y_0) = 0, x_0 = H^{-1}(y_0)$. This w is a non-constant concave function with a maximum at y_0. If w is non-constant to the right and to the left of y_0, then by concavity it will vanish at y_1, y_2 with $y_1 < y_0 < y_2$. The function v will then vanish at $x_1 = H^{-1}(y_1), x_2 = H^{-1}(y_2)$.

Suppose now that w were to remain constant, say, to the right of y_0 (but then non-constant to the left). Then v will reach a zero $x_1 < x_0$ and, at the same time,

$$\int^b v^{-2}dx = \int^b u^{-2}dx = \infty .$$

Thus

$$G(x) = \int_{x_0}^x v^{-2}(t)dt$$

is a function with $SG = 2(q+\epsilon)$ and $G(x_1) = G(b)$ equal to the point at infinity. Because $G : I \to \mathbb{R} \cup \{\infty\}$ is locally injective, it follows that $G(x_2) = G(x_3) = c < \infty$ for some $x_2 > x_1$ and $x_3 < b$. The function

$$\tilde{G} = \frac{1}{G - c}$$

has $S\tilde{G} = SG$ and $\tilde{v} = (\tilde{G}')^{-1/2}$ will be the desired solution with two zeros.

□

We apply Theorem 2.2 with $2q = \psi'' - (1/2)(\psi')^2 + e^{2\psi}\frac{2\pi^2}{\delta^2}$ and H equal to the corresponding extremal. The fact that the modified differential equation admits a non-trivial solution implies the existence of a function $G : I \to \mathbb{R} \cup \{\infty\}$ with $SG = 2q + \epsilon$ which is not injective. This proves the second assertion about sharpness.

3. Other Corollaries

In this section we derive a few other corollaries that we find of particular interest. In all cases, the proof relies on choosing a particular function ψ in Theorem 2.1.

Corollary 3.1. *Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a C^3 curve with nowhere vanishing f' and finite length L. If the curvature k satisfies*

$$k \leq \frac{2\pi}{L}$$

then f is injective.

Proof. We choose $\psi = \log |f'|$, so that $Ss = \psi'' - (1/2)(\psi')^2$. The corollary follows at once.

□

Corollary 3.2. *Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a C^3 curve with nowhere vanishing f'. If*

$$S_1 f \leq 2t\frac{1 + (1-t)x^2}{(1-x^2)^2}\,,\ t \geq 1 \tag{3.1}$$

or

$$S_1 f \leq 2t\frac{1 + (1-t)x^2}{(1-x^2)^2} + \frac{2\pi}{(1-x^2)^2 t}\left(\frac{\Gamma(\frac{3}{2} - t)}{\Gamma(1-t)}\right)^2\,,\ 0 \leq t < 1 \tag{3.2}$$

then f is injective.

Proof. We choose $\psi = -y\log(1-x^2)$. For $t \geq 1$, then $\delta = \infty$, while for $t \in [0,1)$ the diameter is finite and given by

$$\sqrt{\pi}\frac{\Gamma(1-t)}{\Gamma(\frac{3}{2} - t)}\,.$$

□

Inequalities (3.1) and (3.2) represent analogues of criteria for holomorphic mappings derived by Ahlfors [2].

Corollary 3.3. *Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a C^3 curve with nowhere vanishing f'. If σ is a C^2 function on I with*

$$S_1 f + \frac{2x}{1-x^2} \leq \sigma'' - \frac{1}{2}(\sigma')^2 + \frac{2}{(1-x^2)^2}$$

then f is injective.

Corollary 3.3 can be considered an analogue of the Epstein criterion [9].

Proof. We let $\psi = \sigma - \log(1-x^2)$ in Theorem 2.1. □

Corollary 3.4. *Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a C^3 curve with nowhere vanishing f'. If σ is a C^2 function with*

$$\frac{v'}{v}\sigma' + \frac{1}{2}v^2k^2 \leq \sigma'' - \frac{1}{2}(\sigma')^2$$

then f is injective.

Proof. We let $\psi = \log|f'| + \sigma$ in Theorem 2.1. □

Corollary 3.5. *Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a C^3 curve with nowhere vanishing f'. If*

$$\frac{2x}{1-x^2}\frac{v'}{v} + \frac{1}{2}v^2k^2 \leq \frac{2}{(1-x^2)^2}$$

then f is injective.

This final corollary represents an analogue of the condition by Becker [3].

Proof. We let $\psi = \log|f'| - \log(1-x^2)$ in Theorem 2.1. □

4. Distortion and Extensions

The purpose of this final section is to derive the corresponding version of Theorem B for the main result here.

Theorem 4.1. *Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a C^3 curve with nowhere vanishing f' satisfying (2.1). Let $H = F$ or $H = \phi$ be the extremal functions depending on whether condition (2.1) is of infinite or finite type. For $x_0 \in I$ fixed, suppose that*

$$|f'(x_0)| = H'(x_0)\,,\ |f'|'(x_0) = H''(x_0)\,. \tag{4.1}$$

Then

(i) $|f(x_1) - f(x_2)| \leq |H(x_1) - H(x_2)|$ and $|f'(x)| \leq H'(x)$ for all $x, x_1, x_2 \in I$;

(ii) f admits a spherically continuous extension to $\overline{I}$. If f is not injective in the closed interval, the up to a Möbius transformation, $f = H$.

Proof. The normalizations (4.1) are no restriction, in the sense that they can be achieved by composing f with a suitable Möbius transformation.

To show (i), we consider $u = -2\log|f'|$. Then

$$u'' + qu = 0\,,$$

where $2q = Ss$. Because of (2.1), we have that $q \leq (1/2)SH$. The inequality $|f'(x)| \leq H'(x)$ follows at once from Sturm comparison, while the second inequality in (i) follows from integration.

We show the continuous extension, say at $x = b$. Fix $\bar{x}$ near b, and let T be a Möbius transformation so that the curve $g = T(f)$ satisfies

$$|g'(\bar{x})| = G'(\bar{x})\ ,\ |g'|'(\bar{x}) = G''(\bar{x})\,,$$

for $G = -1/H$. If $\bar{x}$ is close to b then G is regular near this endpoint. The previous argument implies that for $x \in (\bar{x}, b)$

$$|g'(x)| \leq G'(x)\,,$$

and

$$|g(x_1) - g(x_2)| \leq |G(x_1) - G(x_2)|$$

for $\bar{x} \leq x_1, x_2 < b$. This shows that the modulus of continuity of g is controlled by that of G near b, and the extension follows.

Suppose now that f is not injective on $\overline{I}$. Since it is injective on I, then either $f(a) = f(b)$ or $f(c) \in \{f(a), f(b)\}$ for some $c \in I$. We claim that the latter cannot occur. Suppose, on the contrary, that say $f(c) = f(b)$ for some $c \in I$. We may assume that this common point lies at infinity. Let $\gamma = H(c)$ and consider the function

$$w(y) = \frac{u}{v}(H^{-1}(y))\,,\ y \in [\gamma, \infty)\,,$$

where $u = |f'|^{-1/2}$ and $v = (H')^{-1/2}$. Then

$$w'' = \frac{1}{2}w^4(SH - Ss) \geq 0\,.$$

If w were constant on $[\gamma, \infty)$ then $|f'|$ would be a constant multiple of H' on $[c, b)$, which would contradict that $f(c)$ is the point at infinity. Hence w cannot be constant, and it is therefor bounded below by some line $my + n, m \neq 0$. If $m > 0$, then the inequality $w(y) \geq my + n > 0$ for large values of y, which would give

$$|f'| \leq \frac{H'}{(mH + n)^2}$$

for all x near b. This estimate implies that

$$\int^b |f'|dx < \infty\,,$$

a contradiction to the fact that $f(b)=\infty$. If $m<0$ we analyze the inequality $w(y)\geq my+n$ for values close to γ, to reach the contradicting conclusion that $f(c)$ is a finite point. This shows that if f is not injective on $\overline{I}$ then $f(a)=f(b)$. Again, after a Möbius shift, we may assume that this common point lies at infinity. We follow the previous argument and the function w to conclude that, up to a constant factor, $|f'|=H'$ on I. But if w is constant, then $SH=Ss$, which implies that the curvature k must vanish identically. Therefore, f traces a straight line at equal speed as H, so $f=H$ up to an isometry. This finishes the proof.

□

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